Optical Waveguide Theory (B)

Manfred Hammer*

Theoretical Electrical Engineering
Paderborn University, Germany

Paderborn University — Summer Semester 2020

* *Theoretical Electrical Engineering, Paderborn University*
Warburger Straße 100, 33098 Paderborn, Germany
Phone: +49(0)5251/60-3560
E-mail: manfred.hammer@uni-paderborn.de

Course overview

Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
- Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
- Oblique semi-guided waves: 2-D integrated optics.
- Summary, concluding remarks.

Vector calculus, keywords

(here: Cartesian coordinates)

Ingredients:

- Space and time coordinates: $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to (x, y, z), \;\; t.$

Vector calculus, keywords

(here: Cartesian coordinates)

Ingredients:

- Space and time coordinates: $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to (x, y, z), \ t.$
- Scalar and vector fields: $\phi(\boldsymbol{r}, t), \ \boldsymbol{A}(\boldsymbol{r}, t), \qquad \boldsymbol{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}.$

Ingredients: (here: Cartesian coordinates)

- Space and time coordinates: $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to (x, y, z),\ t.$
- Scalar and vector fields: $\phi(\boldsymbol{r}, t),\ \boldsymbol{A}(\boldsymbol{r}, t), \qquad \boldsymbol{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}.$
- Inner product: $\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z.$

Vector calculus, keywords

Ingredients:

(here: Cartesian coordinates)

- Space and time coordinates: $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to (x, y, z),\ t.$
- Scalar and vector fields: $\phi(\boldsymbol{r}, t),\ \boldsymbol{A}(\boldsymbol{r}, t),\qquad \boldsymbol{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}.$
- Inner product: $\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z.$
- Vector product: $\boldsymbol{A} \times \boldsymbol{B} = \begin{pmatrix} A_y B_z - A_z B_y \\ A_z B_x - A_x B_z \\ A_x B_y - A_y B_x \end{pmatrix}.$

Vector calculus, keywords

Ingredients: (here: Cartesian coordinates)

- Space and time coordinates: $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to (x, y, z),\ t.$
- Scalar and vector fields: $\phi(\boldsymbol{r}, t),\ \boldsymbol{A}(\boldsymbol{r}, t),\qquad \boldsymbol{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}.$
- Inner product: $\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z.$
- Vector product: $\boldsymbol{A} \times \boldsymbol{B} = \begin{pmatrix} A_y B_z - A_z B_y \\ A_z B_x - A_x B_z \\ A_x B_y - A_y B_x \end{pmatrix}.$
- Time derivatives: $\dfrac{\partial \phi}{\partial t},\ \partial_t \phi,\ \dot{\phi},\ \cancel{\nabla_t \phi}.$

Vector calculus, keywords

(here: Cartesian coordinates)

Ingredients:

- Del, nabla: $\boldsymbol{\nabla} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}.$

Vector calculus, keywords

(here: Cartesian coordinates)

Ingredients:

- Del, nabla: $\boldsymbol{\nabla} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}.$
- Gradient: $\mathrm{grad}\phi = \boldsymbol{\nabla}\phi = \begin{pmatrix} \partial_x \phi \\ \partial_y \phi \\ \partial_z \phi \end{pmatrix}.$

Vector calculus, keywords

(here: Cartesian coordinates)

Ingredients:

- Del, nabla: $\boldsymbol{\nabla} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$.
- Gradient: $\mathrm{grad}\phi = \boldsymbol{\nabla}\phi = \begin{pmatrix} \partial_x\phi \\ \partial_y\phi \\ \partial_z\phi \end{pmatrix}$.
- Divergence: $\mathrm{div}\boldsymbol{A} = \boldsymbol{\nabla} \cdot \boldsymbol{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z$.

Vector calculus, keywords

(here: Cartesian coordinates)

Ingredients:

- Del, nabla: $\boldsymbol{\nabla} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$.
- Gradient: $\mathrm{grad}\phi = \boldsymbol{\nabla}\phi = \begin{pmatrix} \partial_x\phi \\ \partial_y\phi \\ \partial_z\phi \end{pmatrix}$.
- Divergence: $\mathrm{div}\boldsymbol{A} = \boldsymbol{\nabla} \cdot \boldsymbol{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z$.
- Curl: $\mathrm{curl}\boldsymbol{A} = \mathrm{rot}\boldsymbol{A} = \boldsymbol{\nabla} \times \boldsymbol{A} = \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix}$.

(here: Cartesian coordinates)

Ingredients:

- Del, nabla: $\boldsymbol{\nabla} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$.
- Gradient: $\mathrm{grad}\phi = \boldsymbol{\nabla}\phi = \begin{pmatrix} \partial_x\phi \\ \partial_y\phi \\ \partial_z\phi \end{pmatrix}$.
- Divergence: $\mathrm{div}\boldsymbol{A} = \boldsymbol{\nabla}\cdot\boldsymbol{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z$.
- Curl: $\mathrm{curl}\boldsymbol{A} = \mathrm{rot}\boldsymbol{A} = \boldsymbol{\nabla}\times\boldsymbol{A} = \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix}$.
- Laplacian: $\Delta = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} = \boldsymbol{\nabla}^2$,
$$\Delta\phi = \partial_x^2\phi + \partial_y^2\phi + \partial_x^2\phi, \qquad \Delta\boldsymbol{A} = \begin{pmatrix} \Delta A_x \\ \Delta A_y \\ \Delta A_z \end{pmatrix}.$$

Dirac delta

A linear functional
that extracts the value of a function at one point:

1-D: $$\int_a^b f(x)\,\delta(x-x_0)\,\mathrm{d}x = \begin{cases} f(x_0), & \text{if } a < x_0 < b, \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta(x-x_0) = 0, \text{ if } x \neq x_0.$$

Dirac delta

A linear functional
that extracts the value of a function at one point:

1-D: $$\int_a^b f(x)\,\delta(x-x_0)\,\mathrm{d}x = \begin{cases} f(x_0), & \text{if } a < x_0 < b, \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta(x-x_0) = 0, \text{ if } x \neq x_0.$$

3-D: $$\int_{\mathcal{V}} f(\boldsymbol{r})\,\delta(\boldsymbol{r}-\boldsymbol{r}_0)\,\mathrm{d}\mathcal{V} = \begin{cases} f(\boldsymbol{r}_0), & \text{if } \boldsymbol{r}_0 \in \mathcal{V}, \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta(\boldsymbol{r}-\boldsymbol{r}_0) = 0, \text{ if } \boldsymbol{r} \neq \boldsymbol{r}_0.$$

Implications: manifold.

1-D: A function $f(x) \in \mathbb{C}$ of one variable:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k)\, \mathrm{e}^{\mathrm{i}kx} \mathrm{d}k, \qquad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}x.$$

Fourier transform, 1-D

1-D: A function $f(x) \in \mathbb{C}$ of one variable:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k)\ \mathrm{e}^{\mathrm{i}kx} \mathrm{d}k, \qquad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}x.$$

- Arbitrary: positioning of factors $1/\sqrt{2\pi}$, signs of exponents.

Fourier transform, 1-D

1-D: A function $f(x) \in \mathbb{C}$ of one variable:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k)\ \mathrm{e}^{\mathrm{i}kx} \mathrm{d}k, \qquad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}x.$$

- Arbitrary: positioning of factors $1/\sqrt{2\pi}$, signs of exponents.
- $\widetilde{\alpha f_1 + \beta f_2} = \alpha \tilde{f}_1 + \beta \tilde{f}_2$.

Fourier transform, 1-D

1-D: A function $f(x) \in \mathbb{C}$ of one variable:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k)\, \mathrm{e}^{\mathrm{i}kx} \mathrm{d}k, \qquad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}x.$$

- Arbitrary: positioning of factors $1/\sqrt{2\pi}$, signs of exponents.
- $\widetilde{\alpha f_1 + \beta f_2} = \alpha \tilde{f}_1 + \beta \tilde{f}_2$.
- $f(x) = f(-x) \rightsquigarrow \tilde{f}(k) = \tilde{f}(-k)$.

Fourier transform, 1-D

1-D: A function $f(x) \in \mathbb{C}$ of one variable:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k)\, \mathrm{e}^{\mathrm{i}kx} \mathrm{d}k, \qquad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}x.$$

- Arbitrary: positioning of factors $1/\sqrt{2\pi}$, signs of exponents.
- $\widetilde{\alpha f_1 + \beta f_2} = \alpha\tilde{f}_1 + \beta\tilde{f}_2$.
- $f(x) = f(-x) \rightsquigarrow \tilde{f}(k) = \tilde{f}(-k)$.
- $f(x) = -f(-x) \rightsquigarrow \tilde{f}(k) = -\tilde{f}(-k)$.

Fourier transform, 1-D

1-D: A function $f(x) \in \mathbb{C}$ of one variable:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k)\ \mathrm{e}^{\mathrm{i}kx} \mathrm{d}k, \qquad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}x.$$

- Arbitrary: positioning of factors $1/\sqrt{2\pi}$, signs of exponents.
- $\widetilde{\alpha f_1 + \beta f_2} = \alpha \tilde{f}_1 + \beta \tilde{f}_2$.
- $f(x) = f(-x) \rightsquigarrow \tilde{f}(k) = \tilde{f}(-k)$.
- $f(x) = -f(-x) \rightsquigarrow \tilde{f}(k) = -\tilde{f}(-k)$.
- $f \in \mathbb{R} \rightsquigarrow \tilde{f}(-k) = \tilde{f}^*(k)$.

Fourier transform, 1-D

1-D: A function $f(x) \in \mathbb{C}$ of one variable:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k)\, \mathrm{e}^{\mathrm{i}kx} \mathrm{d}k, \qquad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}x.$$

- Arbitrary: positioning of factors $1/\sqrt{2\pi}$, signs of exponents.
- $\widetilde{\alpha f_1 + \beta f_2} = \alpha \tilde{f}_1 + \beta \tilde{f}_2$.
- $f(x) = f(-x) \rightsquigarrow \tilde{f}(k) = \tilde{f}(-k)$.
- $f(x) = -f(-x) \rightsquigarrow \tilde{f}(k) = -\tilde{f}(-k)$.
- $f \in \mathbb{R} \rightsquigarrow \tilde{f}(-k) = \tilde{f}^*(k)$.
- $\delta(x) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} \mathrm{d}k$.

3-D: A field $\phi(\boldsymbol{r})$:

$$\phi(\boldsymbol{r}) = \frac{1}{\sqrt{2\pi}^3} \int \tilde{\phi}(\boldsymbol{k})\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}^3 k, \qquad \tilde{\phi}(\boldsymbol{k}) = \frac{1}{\sqrt{2\pi}^3} \int \phi(\boldsymbol{r})\ \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}^3 r.$$

Fourier transform

3-D: A field $\phi(\boldsymbol{r})$:

$$\phi(\boldsymbol{r}) = \frac{1}{\sqrt{2\pi}^3} \int \tilde{\phi}(\boldsymbol{k})\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}^3 k, \qquad \tilde{\phi}(\boldsymbol{k}) = \frac{1}{\sqrt{2\pi}^3} \int \phi(\boldsymbol{r})\ \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}^3 r.$$

4-D: A field $\phi(\boldsymbol{r}, t)$:

$$\phi(\boldsymbol{r}, t) = \frac{1}{\sqrt{2\pi}^4} \iint \tilde{\phi}(\boldsymbol{k}, \omega)\ \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \mathrm{d}^3 k\, \mathrm{d}\omega,$$

$$\tilde{\phi}(\boldsymbol{k}, \omega) = \frac{1}{\sqrt{2\pi}^4} \iint \phi(\boldsymbol{r}, t)\ \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \mathrm{d}^3 r\, \mathrm{d}t.$$

Directionally constant systems

A linear PDE in two unknowns

$$(A\,\partial_{xx} + B\,\partial_{yy} + C\,\partial_{xy} + D\,\partial_x + E\,\partial_y + F)\,\psi(x,y) = 0,$$

coefficients $A(x,y),\ldots,F(x,y)$.

Directionally constant systems

A linear PDE in two unknowns

$$(A\,\partial_{xx} + B\,\partial_{yy} + C\,\partial_{xy} + D\,\partial_x + E\,\partial_y + F)\,\psi(x,y) = 0,$$

coefficients $A(x,y), \ldots, F(x,y)$.

If the system is constant in x, $\quad \partial_x A = \ldots = \partial_x F = 0\,,$

- write ψ as $\qquad \psi(x,y) = \int \tilde{\psi}(k,y)\ \mathrm{e}^{\mathrm{i}kx}\mathrm{d}k\,.$

A linear PDE in two unknowns

$$(A\,\partial_{xx} + B\,\partial_{yy} + C\,\partial_{xy} + D\,\partial_x + E\,\partial_y + F)\,\psi(x,y) = 0,$$

coefficients $A(x,y),\ldots,F(x,y)$.

If the system is constant in x, $\quad \partial_x A = \ldots = \partial_x F = 0\,,$

- write ψ as $\qquad \psi(x,y) = \int \tilde{\psi}(k,y)\ \mathrm{e}^{\mathrm{i}kx}\mathrm{d}k\,.$

$$\hookrightarrow \quad \int \big(B\,\partial_{yy} + (E + \mathrm{i}kC)\partial_y + (F + \mathrm{i}kD - k^2A)\big)\,\tilde{\psi}(k,y)\ \mathrm{e}^{\mathrm{i}kx}\mathrm{d}k = 0,$$

$$\hookrightarrow \quad \big(B\,\partial_{yy} + (E + \mathrm{i}kC)\partial_y + (F + \mathrm{i}kD - k^2A)\big)\,\tilde{\psi}(k,y) = 0, \ \text{(for all } k\text{)},$$

... a set of DEs in one unknown.

Directionally constant systems

A linear PDE in two unknowns

$$(A\,\partial_{xx} + B\,\partial_{yy} + C\,\partial_{xy} + D\,\partial_x + E\,\partial_y + F)\,\psi(x,y) = 0,$$

coefficients $A(x,y), \ldots, F(x,y)$.

If the system is constant in x, $\quad \partial_x A = \ldots = \partial_x F = 0\,,$

- use an ansatz $\qquad \psi(x,y) = \tilde{\psi}(y)\; \mathrm{e}^{\mathrm{i}kx}\,.$

A linear PDE in two unknowns

$$(A\,\partial_{xx} + B\,\partial_{yy} + C\,\partial_{xy} + D\,\partial_x + E\,\partial_y + F)\,\psi(x,y) = 0,$$

coefficients $A(x,y), \ldots, F(x,y)$.

If the system is constant in x, $\quad \partial_x A = \ldots = \partial_x F = 0\,,$

- use an ansatz $\qquad \psi(x,y) = \tilde{\psi}(y)\ \mathrm{e}^{\mathrm{i}kx}\,.$

$$\hookrightarrow \quad \big(B\,\partial_{yy} + (E + \mathrm{i}kC)\partial_y + (F + \mathrm{i}kD - k^2 A)\big)\,\tilde{\psi}(y) = 0,$$

... a DE in one unknown, with parameter k.

Directionally constant systems

A linear PDE in two unknowns

$$(A\,\partial_{xx} + B\,\partial_{yy} + C\,\partial_{xy} + D\,\partial_x + E\,\partial_y + F)\,\psi(x,y) = 0,$$

coefficients $A(x,y), \ldots, F(x,y)$.

If the system is constant in x, $\partial_x A = \ldots = \partial_x F = 0\,,$

- use an ansatz $\psi(x,y) = \tilde{\psi}(y)\ \mathrm{e}^{\mathrm{i}kx}\,.$

$$\hookrightarrow \quad \big(B\,\partial_{yy} + (E + \mathrm{i}kC)\partial_y + (F + \mathrm{i}kD - k^2A)\big)\,\tilde{\psi}(y) = 0,$$

… a DE in one unknown, with parameter k.

(& boundary conditions, …)

General solution of the wave equation

$$\left(\Delta - \frac{1}{\mathrm{c}^2}\frac{\partial^2}{\partial t^2}\right)\psi(\boldsymbol{r},t) = 0, \quad \psi(\boldsymbol{r},0) = \psi_0(\boldsymbol{r}), \quad \partial_t\psi(\boldsymbol{r},0) = \phi_0(\boldsymbol{r}),$$

General solution of the wave equation

$$\left(\Delta - \frac{1}{\mathrm{c}^2}\frac{\partial^2}{\partial t^2}\right)\psi(\boldsymbol{r},t) = 0, \quad \psi(\boldsymbol{r},0) = \psi_0(\boldsymbol{r}), \quad \partial_t\psi(\boldsymbol{r},0) = \phi_0(\boldsymbol{r}),$$

& $$\psi(\boldsymbol{r},t) = \frac{1}{(2\pi)^2}\iint \tilde{\psi}(\boldsymbol{k},\omega)\,\mathrm{e}^{\mathrm{i}\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t\right)}\,\mathrm{d}\omega\,\mathrm{d}^3k,$$

General solution of the wave equation

$$\left(\Delta - \frac{1}{\mathrm{c}^2}\frac{\partial^2}{\partial t^2}\right)\psi(\boldsymbol{r},t) = 0, \quad \psi(\boldsymbol{r},0) = \psi_0(\boldsymbol{r}), \quad \partial_t\psi(\boldsymbol{r},0) = \phi_0(\boldsymbol{r}),$$

& $$\psi(\boldsymbol{r},t) = \frac{1}{(2\pi)^2}\iint \tilde{\psi}(\boldsymbol{k},\omega)\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\,\mathrm{d}\omega\,\mathrm{d}^3k,$$

$$\hookrightarrow \quad \left(-\boldsymbol{k}^2 + \frac{\omega^2}{\mathrm{c}^2}\right)\tilde{\psi}(\boldsymbol{k},\omega) = 0,$$

General solution of the wave equation

$$\left(\Delta - \frac{1}{\mathrm{c}^2}\frac{\partial^2}{\partial t^2}\right)\psi(\boldsymbol{r},t) = 0, \quad \psi(\boldsymbol{r},0) = \psi_0(\boldsymbol{r}), \quad \partial_t\psi(\boldsymbol{r},0) = \phi_0(\boldsymbol{r}),$$

& $$\psi(\boldsymbol{r},t) = \frac{1}{(2\pi)^2}\iint \tilde{\psi}(\boldsymbol{k},\omega)\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\,\mathrm{d}\omega\,\mathrm{d}^3k,$$

$\hookrightarrow$ $$\left(-\boldsymbol{k}^2 + \frac{\omega^2}{\mathrm{c}^2}\right)\tilde{\psi}(\boldsymbol{k},\omega) = 0,$$

$\hookrightarrow$ $$\tilde{\psi}(\boldsymbol{k},\omega) = a_{\mathrm{f}}(\boldsymbol{k})\,\delta(\omega-\omega_k) + a_{\mathrm{b}}(\boldsymbol{k})\,\delta(\omega+\omega_k), \qquad \omega_k = \mathrm{c}\,|\boldsymbol{k}|,$$

General solution of the wave equation

$$\left(\Delta - \frac{1}{\mathrm{c}^2}\frac{\partial^2}{\partial t^2}\right)\psi(\boldsymbol{r},t) = 0, \quad \psi(\boldsymbol{r},0) = \psi_0(\boldsymbol{r}), \quad \partial_t\psi(\boldsymbol{r},0) = \phi_0(\boldsymbol{r}),$$

& $$\psi(\boldsymbol{r},t) = \frac{1}{(2\pi)^2}\iint \tilde{\psi}(\boldsymbol{k},\omega)\,\mathrm{e}^{\mathrm{i}\,(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}\,\mathrm{d}\omega\,\mathrm{d}^3k,$$

$$\hookrightarrow \quad \left(-\boldsymbol{k}^2 + \frac{\omega^2}{\mathrm{c}^2}\right)\tilde{\psi}(\boldsymbol{k},\omega) = 0,$$

$$\hookrightarrow \quad \tilde{\psi}(\boldsymbol{k},\omega) = a_{\mathrm{f}}(\boldsymbol{k})\,\delta(\omega - \omega_k) + a_{\mathrm{b}}(\boldsymbol{k})\,\delta(\omega + \omega_k), \qquad \omega_k = \mathrm{c}\,|\boldsymbol{k}|,$$

$$\hookrightarrow \quad \psi(\boldsymbol{r},t) = \frac{1}{(2\pi)^2}\int\left(a_{\mathrm{f}}(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}\,(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_k t)} + a_{\mathrm{b}}(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}\,(\boldsymbol{k}\cdot\boldsymbol{r} + \omega_k t)}\right)\mathrm{d}^3k,$$

General solution of the wave equation

$$\left(\Delta - \frac{1}{\mathrm{c}^2}\frac{\partial^2}{\partial t^2}\right)\psi(\boldsymbol{r},t) = 0, \quad \psi(\boldsymbol{r},0) = \psi_0(\boldsymbol{r}), \quad \partial_t\psi(\boldsymbol{r},0) = \phi_0(\boldsymbol{r}),$$

& $$\psi(\boldsymbol{r},t) = \frac{1}{(2\pi)^2}\iint \tilde{\psi}(\boldsymbol{k},\omega)\,\mathrm{e}^{\mathrm{i}\,(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}\,\mathrm{d}\omega\,\mathrm{d}^3k,$$

$\hookrightarrow$ $$\left(-\boldsymbol{k}^2 + \frac{\omega^2}{\mathrm{c}^2}\right)\tilde{\psi}(\boldsymbol{k},\omega) = 0,$$

$\hookrightarrow$ $$\tilde{\psi}(\boldsymbol{k},\omega) = a_\mathrm{f}(\boldsymbol{k})\,\delta(\omega - \omega_k) + a_\mathrm{b}(\boldsymbol{k})\,\delta(\omega + \omega_k), \qquad \omega_k = \mathrm{c}\,|\boldsymbol{k}|,$$

$\hookrightarrow$ $$\psi(\boldsymbol{r},t) = \frac{1}{(2\pi)^2}\int\left(a_\mathrm{f}(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}\,(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_k t)} + a_\mathrm{b}(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}\,(\boldsymbol{k}\cdot\boldsymbol{r} + \omega_k t)}\right)\mathrm{d}^3k,$$

- $\psi(\boldsymbol{r},0) = \psi_0(\boldsymbol{r}),\ \partial_t\psi(\boldsymbol{r},0) = \phi_0(\boldsymbol{r}) \rightsquigarrow \ldots \rightsquigarrow a_\mathrm{f}(\boldsymbol{k}), a_\mathrm{b}(\boldsymbol{k}).$

- Functional:
$$\begin{array}{rccc} \mathcal{L}: & U & \longrightarrow & \mathbb{R}, \mathbb{C}, \\ & u & \longrightarrow & \mathcal{L}(u), \end{array}$$

  a map from a space U of functions to real / complex numbers.

- Functional:
$$\begin{aligned} \mathcal{L}: \quad U &\longrightarrow \mathbb{R}, \mathbb{C}, \\ u &\longrightarrow \mathcal{L}(u), \end{aligned}$$
a map from a space U of functions to real / complex numbers.

- Stationary functional:
$$\frac{\mathrm{d}}{\mathrm{d}s} \mathcal{L}(u + s\,v)\Big|_{s=0} = 0 \quad \text{for all} \;\; v,$$
the variation of $\mathcal{L}$ at u vanishes for arbitrary directions v.

A touch of variational calculus

- Functional: $$\begin{array}{rccl} \mathcal{L}: & U & \longrightarrow & \mathbb{R}, \mathbb{C}, \\ & u & \longrightarrow & \mathcal{L}(u), \end{array}$$

  a map from a space U of functions to real / complex numbers.

- Stationary functional: $$\frac{\mathrm{d}}{\mathrm{d}s} \mathcal{L}(u + s\,v)\Big|_{s=0} = 0 \quad \text{for all} \;\; v,$$

  the variation of $\mathcal{L}$ at u vanishes for arbitrary directions v.

- Restriction of a functional:

  ... to a parametrized family of functions;

  $\leftrightsquigarrow$ extremization with respect to these parameters,
  $\leftrightsquigarrow$ approximations of stationary points of the functional.

A touch of variational calculus

Example:

$$U = \{u : [0,\pi] \to \mathbb{R} \mid u(0) = u(\pi) = 0\},$$

$$\mathcal{L} : U \to \mathbb{R}, \qquad \mathcal{L}(u) = \frac{\displaystyle\int_0^\pi (\partial_x u)^2 \mathrm{d}x}{\displaystyle\int_0^\pi u^2 \mathrm{d}x}.$$

(. . .)

Example:

$$U = \{u : [0,\pi] \to \mathbb{R} \mid u(0) = u(\pi) = 0\},$$

$$\mathcal{L} : U \to \mathbb{R}, \qquad \mathcal{L}(u) = \frac{\displaystyle\int_0^\pi (\partial_x u)^2 \mathrm{d}x}{\displaystyle\int_0^\pi u^2 \mathrm{d}x}.$$

(. . .)

$\mathcal{L}$ stationary at u,

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}(u + s\,v)\Big|_{s=0} = 0 \quad \forall\, v.$$

$\leftrightsquigarrow$

u satisfies DE & b.c.,

$$\partial_x^2 u = -\lambda u, \quad \lambda = \mathcal{L}(u),$$

$$u(0) = u(\pi) = 0.$$

Example:

$U = \{u : [0,\pi] \to \mathbb{R} \mid u(0) = u(\pi) = 0\}$,

$\mathcal{L} : U \to \mathbb{R}$,

$$\mathcal{L}(u) = \frac{\displaystyle\int_0^\pi (\partial_x u)^2 \mathrm{d}x}{\displaystyle\int_0^\pi u^2 \mathrm{d}x}.$$

(. . .)

$\mathcal{L}$ stationary at u,

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}(u + s\,v)\Big|_{s=0} = 0 \quad \forall\, v.$$

$\leftrightsquigarrow$

u satisfies DE & b.c.,

$\partial_x^2 u = -\lambda u, \quad \lambda = \mathcal{L}(u)$,

$u(0) = u(\pi) = 0$.

$\downarrow$ Restrict $\mathcal{L}$, $\quad L(\boldsymbol{a}) = \mathcal{L}(u|_{\boldsymbol{a}})$.

L stationary at $\boldsymbol{a}$, $\quad \boldsymbol{\nabla}_{\boldsymbol{a}} L = 0$. $\leftrightsquigarrow$ Approximate solution of DE / eigenproblem.

Upcoming

Next lectures:

- Maxwell equations, different formulations, interfaces, energy and power flow.
- Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- Normal modes of dielectric optical waveguides, mode interference.