

Optical Waveguide Theory (B)



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Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B **Brush up on mathematical tools.**
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
- J A touch of photonic crystals; a touch of plasmonics.
 - Hybrid analytical / numerical coupled mode theory.
 - Oblique semi-guided waves: 2-D integrated optics.

Vector calculus, keywords

Ingredients:

(here: Cartesian coordinates)

- Space and time coordinates: $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow (x, y, z), t.$

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- Time derivatives: $\frac{\partial \phi}{\partial t}, \partial_t \phi, \dot{\phi}, \nabla_T \phi$.

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- Curl: $\text{curl}\mathbf{A} = \text{rot}\mathbf{A} = \nabla \times \mathbf{A} = \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix}$.

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- Laplacian: $\Delta = \nabla \cdot \nabla = \nabla^2$,
 $\Delta\phi = \partial_x^2\phi + \partial_y^2\phi + \partial_z^2\phi$, $\Delta\mathbf{A} = \begin{pmatrix} \Delta A_x \\ \Delta A_y \\ \Delta A_z \end{pmatrix}$.

Dirac delta

A linear functional
that extracts the value of a function at one point:



$$\mathbf{1-D:} \quad \int_a^b f(x) \delta(x - x_0) dx = \begin{cases} f(x_0), & \text{if } a < x_0 < b, \\ 0 & \text{otherwise;} \end{cases}$$
$$\delta(x - x_0) = 0, \text{ if } x \neq x_0.$$

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$$3\text{-D: } \int_{\mathcal{V}} f(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}_0) d\mathcal{V} = \begin{cases} f(\mathbf{r}_0), & \text{if } \mathbf{r}_0 \in \mathcal{V}, \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta(\mathbf{r} - \mathbf{r}_0) = 0, \text{ if } \mathbf{r} \neq \mathbf{r}_0.$$

Implications: manifold.

Fourier transform, 1-D

1-D: A function $f(x) \in \mathbb{C}$ of one variable:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk, \quad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx.$$

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- $\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk$.

Fourier transform

3-D: A field $\phi(\mathbf{r})$:

$$\phi(\mathbf{r}) = \frac{1}{\sqrt{2\pi}^3} \int \tilde{\phi}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}} d^3k, \quad \tilde{\phi}(\mathbf{k}) = \frac{1}{\sqrt{2\pi}^3} \int \phi(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} d^3r.$$

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4-D: A field $\phi(\mathbf{r}, t)$:

$$\phi(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}^4} \iiint \tilde{\phi}(\mathbf{k}, \omega) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} d^3k d\omega,$$
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Directionally constant systems

A **linear** PDE in two unknowns

$$(A \partial_{xx} + B \partial_{yy} + C \partial_{xy} + D \partial_x + E \partial_y + F) \psi(x, y) = 0,$$

coefficients $A(x, y), \dots, F(x, y)$.

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↪ $\int (B \partial_{yy} + (E + ikC) \partial_y + (F + ikD - k^2 A)) \tilde{\psi}(k, y) e^{ikx} dk = 0,$

↪ $(B \partial_{yy} + (E + ikC) \partial_y + (F + ikD - k^2 A)) \tilde{\psi}(k, y) = 0, \text{ (for all } k),$

... a set of DEs in one unknown.

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(& boundary conditions, ...)

General solution of the wave equation

$$\left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \psi(\mathbf{r}, t) = 0, \quad \psi(\mathbf{r}, 0) = \psi_0(\mathbf{r}), \quad \partial_t \psi(\mathbf{r}, 0) = \phi_0(\mathbf{r}),$$

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A touch of variational calculus

- **Functional:**
$$\begin{aligned}\mathcal{L} : U &\longrightarrow \mathbb{R}, \mathbb{C}, \\ u &\longrightarrow \mathcal{L}(u),\end{aligned}$$

a map from a space U of functions to real / complex numbers.

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- **Stationary** functional:
$$\left. \frac{d}{ds} \mathcal{L}(u + s v) \right|_{s=0} = 0 \quad \text{for all } v,$$

the variation of \mathcal{L} at u vanishes for arbitrary directions v .

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- **Restriction of a functional:**

... to a parametrized family of functions;

↔ extremization with respect to these parameters,

↔ approximations of stationary points of the functional.

A touch of variational calculus

Example:

$$U = \{u : [0, \pi] \rightarrow \mathbb{R} \mid u(0) = u(\pi) = 0\},$$

$$\mathcal{L} : U \rightarrow \mathbb{R},$$

(...)

$$\mathcal{L}(u) = \frac{\int_0^\pi (\partial_x u)^2 dx}{\int_0^\pi u^2 dx}.$$

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u satisfies DE & b.c.,

$$\begin{aligned} \partial_x^2 u &= -\lambda u, \quad \lambda = \mathcal{L}(u), \\ u(0) &= u(\pi) = 0. \end{aligned}$$

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$$\frac{d}{ds} \mathcal{L}(u + sv) \Big|_{s=0} = 0 \quad \forall v. \quad \longleftrightarrow$$

\downarrow Restrict \mathcal{L} , $L(\mathbf{a}) = \mathcal{L}(u|\mathbf{a})$.

$$L \text{ stationary at } \mathbf{a}, \quad \nabla_{\mathbf{a}} L = 0. \quad \longleftrightarrow$$

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Approximate solution
of DE / eigenproblem.

Upcoming

Next lectures:

- Maxwell equations, different formulations, interfaces, energy and power flow.
- Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- Normal modes of dielectric optical waveguides, mode interference.

