

# *Optical Waveguide Theory (G)*



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*Paderborn University — Summer Semester 2026*

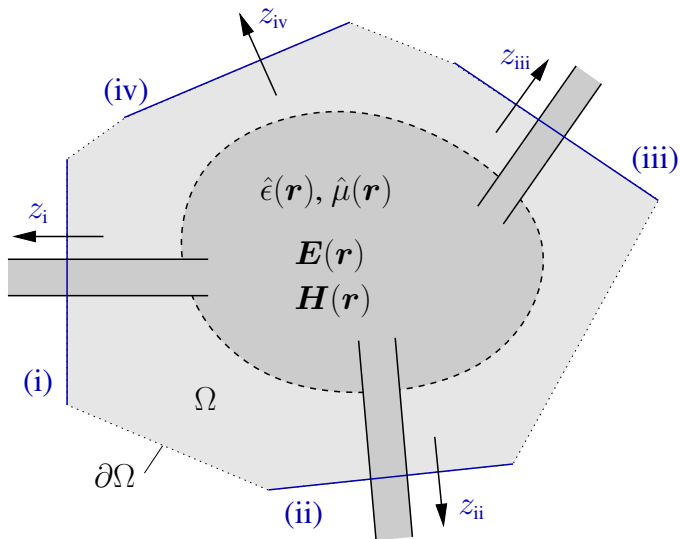
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### Optical waveguide theory

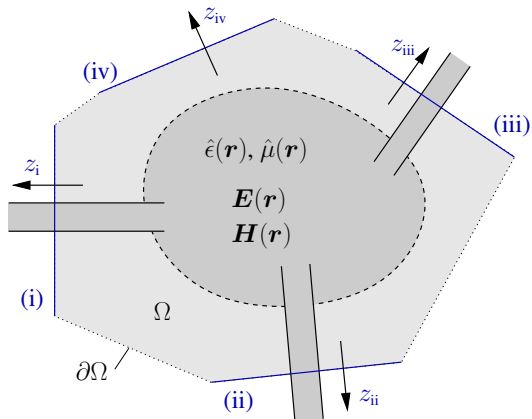
- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
- J A touch of photonic crystals; a touch of plasmonics.
  - Hybrid analytical / numerical coupled mode theory.
  - Oblique semi-guided waves: 2-D integrated optics.

## PICs, OICs, scattering matrices



## Scattering matrices, prerequisites

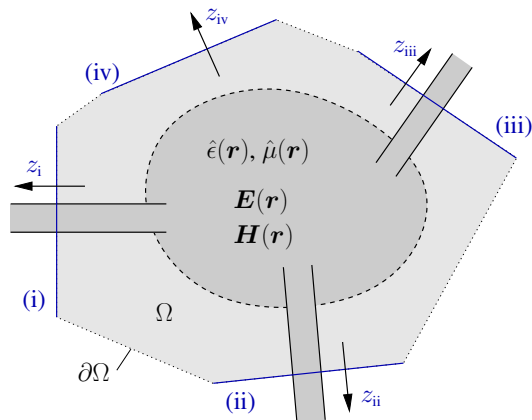
$\sim \exp(i\omega t)$  (FD)



- Passive, linear circuit.

## Scattering matrices, prerequisites

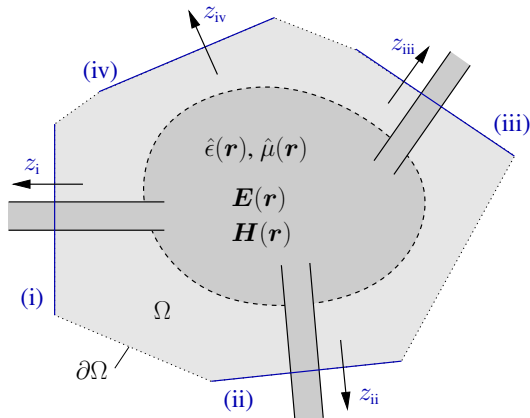
$\sim \exp(i\omega t)$  (FD)



- (Computational) domain of interest  $\Omega$ , its boundary  $\partial\Omega$ .

## Scattering matrices, prerequisites

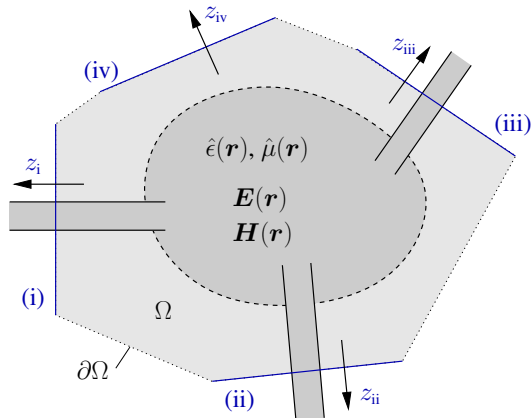
$\sim \exp(i\omega t)$  (FD)



- Connecting channels: lossless waveguides (or “half-spaces”).

## Scattering matrices, prerequisites

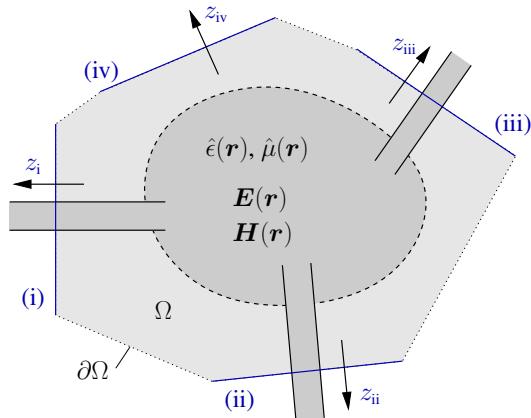
$\sim \exp(i\omega t)$  (FD)



- Physical ports  $p = i, ii, \dots$ : waveguide cross-section planes, local coordinates  $x_p, y_p, z_p$ ; local axis  $z_p$  oriented outwards of  $\Omega$ .

## Scattering matrices, prerequisites

$\sim \exp(i\omega t)$  (FD)

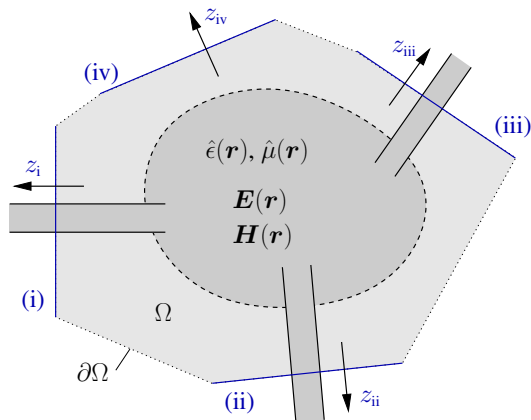


- Establish sets  $\mathcal{N}_p$  of *propagating* directional normal modes  $\{\psi_{p,m}^d := (\mathbf{E}_{p,m}^d, \mathbf{H}_{p,m}^d), \beta_{p,m}; d = \mathbf{f}, \mathbf{b}\}$  on each port  $p$ .

(Restriction to propagating fields: a condition on port positioning / a model assumption.)

## Scattering matrices, prerequisites

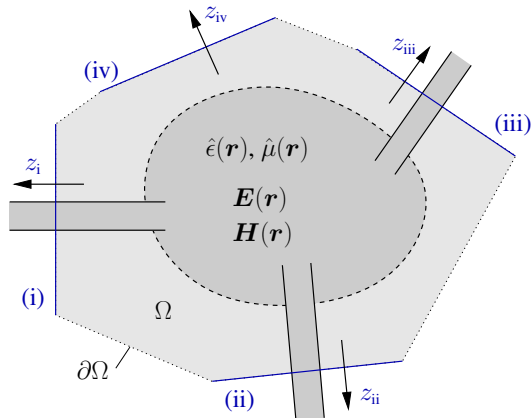
$\sim \exp(i\omega t)$  (FD)



- Ports & modes are such that all mode fields vanish
  - on all “other” port planes, and
  - on  $\partial\Omega$  outside the ports.

## Scattering matrices, prerequisites

$\sim \exp(i\omega t)$  (FD)



Field on port plane  $p$  and “outside”:

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (x_p, y_p, z_p) = \sum_{m \in \mathcal{N}_p} F_{p,m} \psi_{p,m}^f(x_p, y_p) e^{-i\beta_{p,m} z_p} + B_{p,m} \psi_{p,m}^b(x_p, y_p) e^{i\beta_{p,m} z_p}$$

## Scattering matrices, prerequisites

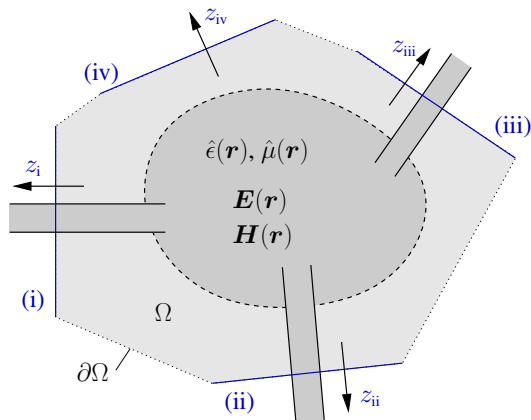
- Passive, linear circuit.  $\sim \exp(i\omega t)$  (FD)
- (Computational) domain of interest  $\Omega$ , its boundary  $\partial\Omega$ .
- Connecting channels: lossless waveguides (or “half-spaces”).
- Physical **ports**  $p = \text{i, ii}, \dots$ : waveguide cross-section planes, local coordinates  $x_p, y_p, z_p$ ; local axis  $z_p$  oriented outwards of  $\Omega$ .
- Establish sets  $\mathcal{N}_p$  of *propagating* directional normal modes  $\{\psi_{p,m}^d := (\mathbf{E}_{p,m}^d, \mathbf{H}_{p,m}^d), \beta_{p,m}; d = \text{f,b}\}$  on each port  $p$ .  
(Restriction to propagating fields: a condition on port positioning / a model assumption.)
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## Scattering matrices

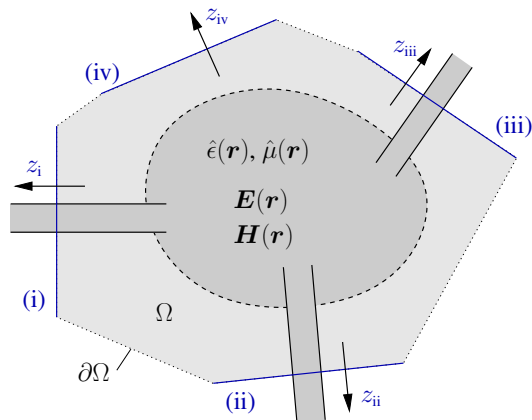
$\sim \exp(i\omega t)$  (FD)



- Merge all mode indices  $\{m\}$  and port IDs  $\{p\}$  into one set of mode identifiers  $\{\nu\}$ ,  $\mathcal{N} = \cup_p \mathcal{N}_p$ .

## Scattering matrices

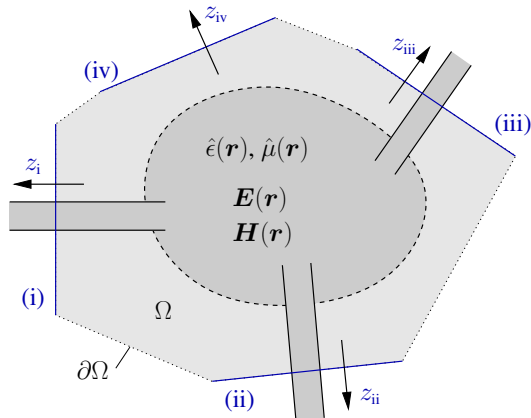
$\sim \exp(i\omega t)$  (FD)



- Assert that  $\psi_{p,\cdot}(\mathbf{r}) = 0$  for all  $\mathbf{r} \in \partial\Omega$ ,  $\mathbf{r} \notin \text{port } p$ .

# Scattering matrices

$\sim \exp(i\omega t)$  (FD)

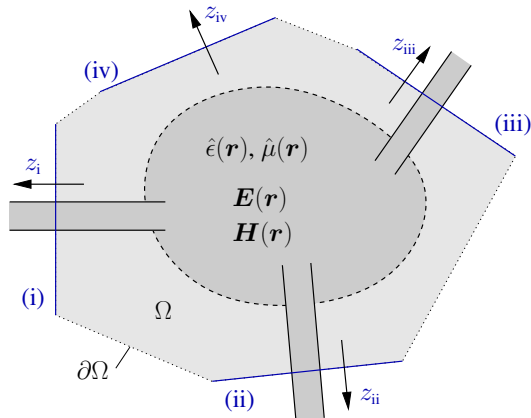


- Field on  $\partial\Omega$ : 
$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \sum_{\nu \in \mathcal{N}} \{F_{\nu} \psi_{\nu}^f + B_{\nu} \psi_{\nu}^b\}.$$

(Position arguments omitted.)

## Scattering matrices

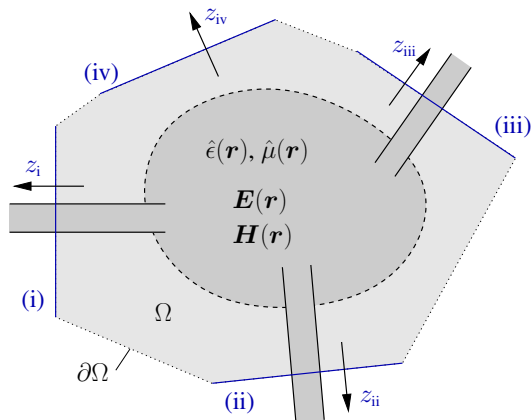
$\sim \exp(i\omega t)$  (FD)



- $B_\nu$ :  $\sim$  incident modes, traveling towards the interior of  $\Omega$ .  
 $F_\nu$ :  $\sim$  outgoing modes, traveling towards the exterior of  $\Omega$ .  
Combine into amplitude vectors  $\mathbf{B}, \mathbf{F}$ .

## Scattering matrices

$\sim \exp(i\omega t)$  (FD)

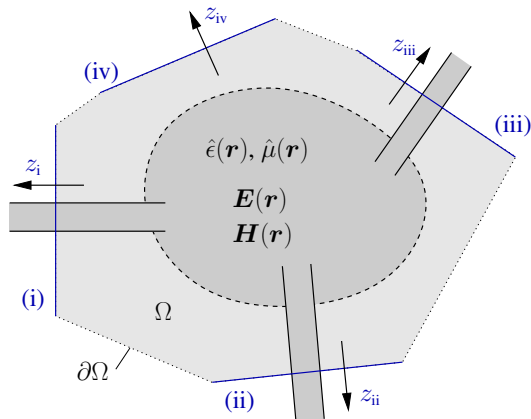


Linear circuit  $\longleftrightarrow$  linear dependence of  $\mathbf{F}$  on  $\mathbf{B}$ ,

Scattering matrix  $\mathbf{S}$  of the circuit:  $\mathbf{F} = \mathbf{S}\mathbf{B}$ ,  $\mathbf{S} = (S_{\nu\mu})$ .

## Scattering matrices

$\sim \exp(i\omega t)$  (FD)



- $S_{\nu\nu}$ :  $\sim (\nu, \mathbf{b}) \rightarrow (\nu, \mathbf{f})$ , reflection coefficient for mode  $\nu$ .
- $S_{\nu\mu}$ :  $\sim (\mu, \mathbf{b}) \rightarrow (\nu, \mathbf{f})$ , transmission coefficient for modes  $\mu, \nu$ .

## Scattering matrices

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- Merge all mode indices  $\{m\}$  and port IDs  $\{p\}$  into one set of mode identifiers  $\{\nu\}$ ,  $\mathcal{N} = \cup_p \mathcal{N}_p$ .  $\sim \exp(i\omega t)$  (FD)
  - Assert that  $\psi_{p,\cdot}(\mathbf{r}) = 0$  for all  $\mathbf{r} \in \partial\Omega$ ,  $\mathbf{r} \notin \text{port } p$ .
  - Field on  $\partial\Omega$ : 
$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \sum_{\nu \in \mathcal{N}} \{F_\nu \psi_\nu^f + B_\nu \psi_\nu^b\}.$$
 (Position arguments omitted.)
  - $B_\nu$ :  $\sim$  **incident modes**, traveling towards the interior of  $\Omega$ .  
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- 

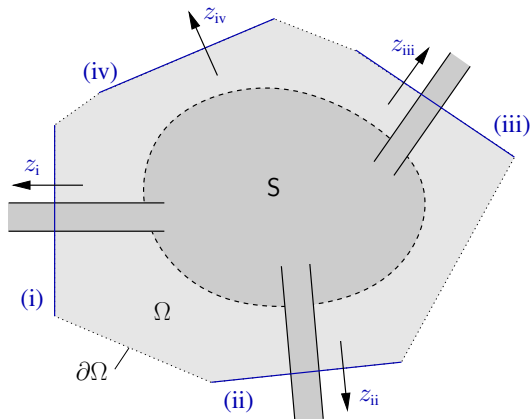
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**Scattering matrix**  $\mathbf{S}$  of the circuit:  $\mathbf{F} = \mathbf{S}\mathbf{B}$ ,  $\mathbf{S} = (S_{\nu\mu})$ .

- $S_{\nu\nu}$ :  $\sim (\nu, b) \rightarrow (\nu, f)$ , **reflection coefficient** for mode  $\nu$ .
- $S_{\nu\mu}$ :  $\sim (\mu, b) \rightarrow (\nu, f)$ , **transmission coefficient** for modes  $\mu, \nu$ .

## PICs, OICs, scattering matrices, scenarios

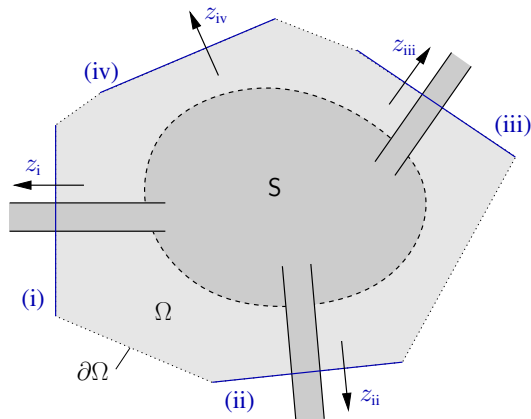
$\sim \exp(i\omega t)$  (FD)



- Scenario: Full matrix  $\mathbf{S}$ , including guided and radiation modes, large  $\dim \mathbf{S} \leftrightarrow$  theoretical results.

## PICs, OICs, scattering matrices, scenarios

$\sim \exp(i\omega t)$  (FD)

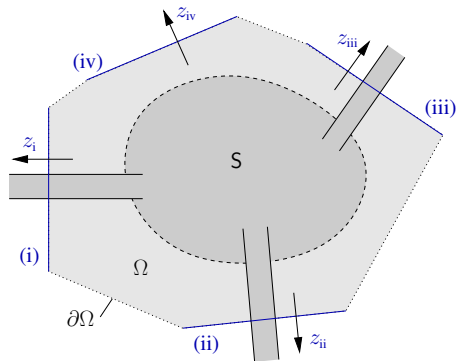


- Scenario: Restrict to a specific set of (guided) modes, or:  
Only a small set of guided modes are relevant:  
small  $\dim \mathbf{S} = N \times N \leftrightarrow$  an  $N$ -port circuit, a  $2$ - $N$ -pole.

( $N$ : the total number of relevant modes, not the number of ports.)

## Scattering matrices, port plane positions

$\sim \exp(i\omega t)$  (FD)



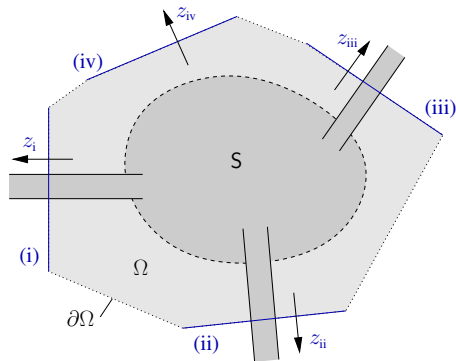
- Shift port plane of mode  $\nu$  by  $\Delta z_\nu$ :  $F_\nu \rightarrow F'_\nu = F_\nu e^{-i\beta_\nu \Delta z_\nu}$ ,  
 Shift port plane of mode  $\mu$  by  $\Delta z_\mu$ :  $B_\mu \rightarrow B'_\mu = B_\mu e^{i\beta_\mu \Delta z_\mu}$ ,  
 $\hookrightarrow F'_\nu = S'_{\nu\mu} B'_\mu$ ,  $S'_{\nu\mu} = S_{\nu\mu} e^{-i(\beta_\nu \Delta z_\nu + \beta_\mu \Delta z_\mu)}$ .

(Moving port planes  $\leftrightarrow$  Phase change in reflection/transmission coefficients.)

(Moving port planes  $\leftrightarrow$  No effect on reflectances/transmittances.)

## Scattering matrices, port mode orthogonality

$\sim \exp(i\omega t)$  (FD)



- Orthogonality relations on port plane  $p$ :

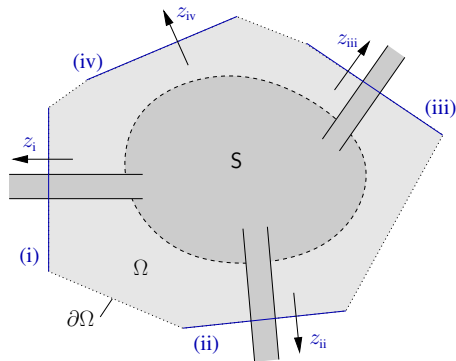
$$(\mathbf{E}_a, \mathbf{H}_a; \mathbf{E}_b, \mathbf{H}_b) = \frac{1}{4} \iint_p (E_{ax}^* H_{by} - E_{ay}^* H_{bx} + H_{ay}^* E_{bx} - H_{ax}^* E_{by}) dx_p dy_p$$

$$(\psi_{p,l}^d; \psi_{p,m}^r) = \pm \delta_{dr} \delta_{lm} P_{p,m}$$

(Things restricted to propagating modes.)

## Scattering matrices, port mode orthogonality

$\sim \exp(i\omega t)$  (FD)



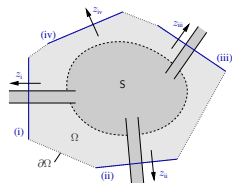
- Extend to the full boundary  $\partial\Omega$ :

$$(\mathbf{E}_a, \mathbf{H}_a; \mathbf{E}_b, \mathbf{H}_b) := \frac{1}{4} \int_{\partial\Omega} (\mathbf{E}_a^* \times \mathbf{H}_b + \mathbf{E}_b \times \mathbf{H}_a^*) \cdot \mathbf{d}\mathbf{a}$$

$$\hookrightarrow (\psi_{p,l}^d; \psi_{q,m}^r) = \pm \delta_{dr} \delta_{pq} \delta_{lm} P_{p,m} \quad \text{or} \quad (\psi_\nu^d; \psi_\mu^r) = \pm \delta_{dr} \delta_{\nu\mu} P_\nu.$$

(Modes belonging to different ports are mutually orthogonal.)

## Scattering matrices, power balance

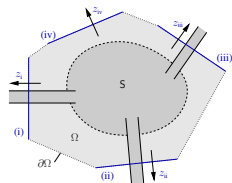


$\sim \exp(i\omega t)$  (FD)

- Net power outflow across the border of the circuit:

$$P = \int_{\partial\Omega} \mathbf{S} \cdot d\mathbf{a} = (\mathbf{E}, \mathbf{H}; \mathbf{E}, \mathbf{H}) = \sum_p \sum_{m \in \mathcal{N}_p} (|F_{p,m}|^2 - |B_{p,m}|^2) P_{p,m}$$

## Scattering matrices, power balance

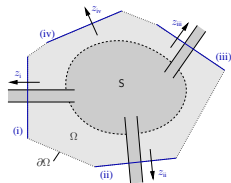


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## Scattering matrices, power balance



$\sim \exp(i\omega t)$  (FD)

- Net power outflow across the border of the circuit:

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 \end{aligned}$$

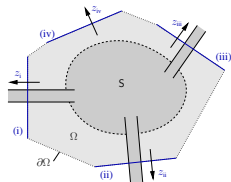
$|B_\mu|^2 P_\mu$ : incident power carried by mode  $\mu$ ,

$|F_\nu|^2 P_\nu$ : outgoing power carried by mode  $\nu$ ,

$(B_\xi = 0 \quad \forall \xi \neq \mu)$

$F_\nu = S_{\nu\mu} B_\mu$ .

## Scattering matrices, power balance



$\sim \exp(i\omega t)$  (FD)

- Net power outflow across the border of the circuit:

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 &= \sum_{\nu \in \mathcal{N}} (|F_\nu|^2 - |B_\nu|^2) P_\nu,
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$|B_\mu|^2 P_\mu$ : incident power carried by mode  $\mu$ ,

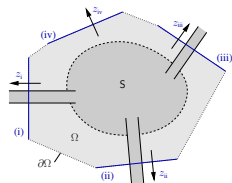
$|F_\nu|^2 P_\nu$ : outgoing power carried by mode  $\nu$ ,

$$F_\nu = S_{\nu\mu} B_\mu.$$

$$|S_{\nu\mu}|^2 \frac{P_\nu}{P_\mu} = \frac{|F_\nu|^2 P_\nu}{|B_\mu|^2 P_\mu}, \quad \begin{array}{l} \mu \neq \nu: \text{ power transmittance } \mu \rightarrow \nu, \\ \mu = \nu: \text{ power reflectance for mode } \nu. \end{array}$$

(Uniform normalized modes,  $P_\nu = P_\mu$ : transmittances are directly given by elements of the scattering matrix).

## Scattering matrices, power balance



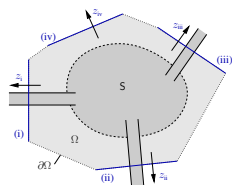
$\sim \exp(i\omega t)$  (FD)

- Net power outflow across the border of the circuit:

$$P = \int_{\partial\Omega} \mathbf{S} \cdot d\mathbf{a} = (\mathbf{E}, \mathbf{H}; \mathbf{E}, \mathbf{H}) = P_0 \left( \mathbf{B}^* \cdot (\mathbf{S}^\dagger \mathbf{S} - \mathbf{1}) \mathbf{B} \right),$$

uniform normalization,  $P_\nu = P_0$  for all  $\nu$ .

## Scattering matrices, power balance



$\sim \exp(i\omega t)$  (FD)

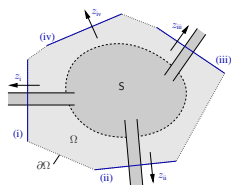
- Net power outflow across the border of the circuit:

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uniform normalization,  $P_\nu = P_0$  for all  $\nu$ .

- Lossless circuit  $\iff \int_{\partial\Omega} \mathbf{S} \cdot d\mathbf{a} = 0 \iff \mathbf{S}^\dagger \mathbf{S} = \mathbf{1}$ ,  
the scattering matrix of a lossless circuit is unitary.

## Scattering matrices, power balance



$\sim \exp(i\omega t)$  (FD)

- Net power outflow across the border of the circuit:

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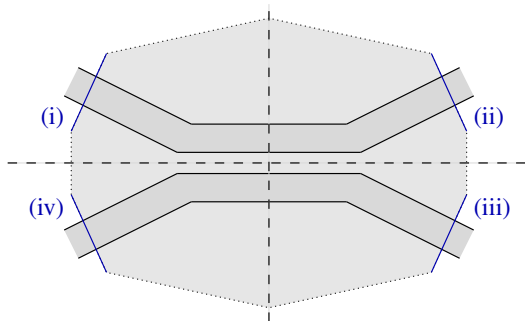
- Lossless circuit  $\iff \int_{\partial\Omega} \mathbf{S} \cdot d\mathbf{a} = 0 \iff \mathbf{S}^\dagger \mathbf{S} = \mathbf{1}$ ,  
the scattering matrix of a lossless circuit is unitary.

- Lossy circuit  $\iff \int_{\partial\Omega} \mathbf{S} \cdot d\mathbf{a} \leq 0 \iff \mathbf{B}^* \cdot \mathbf{S}^\dagger \mathbf{S} \mathbf{B} \leq \mathbf{B}^* \mathbf{B}$ ,

$$\sum_{\nu} |S_{\nu\mu}|^2 \leq 1 \text{ for all } \mu. \quad (\text{The sum of transmittances mode } \mu \text{ to all other modes } \nu \text{ is less than one.})$$

(Interior lossy media, or radiative losses: outgoing propagating modes not taken into account.)

## Scattering matrices, symmetry

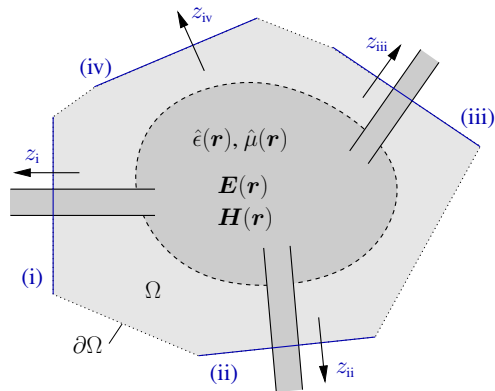


Circuit with specific spatial symmetry  
& symmetrical setting of the port planes

↪ respective symmetry in related coefficients of  $\mathbf{S}$ ,  
symmetric power transmission properties.

## Scattering matrices, reciprocity

$\sim \exp(i\omega t)$  (FD)

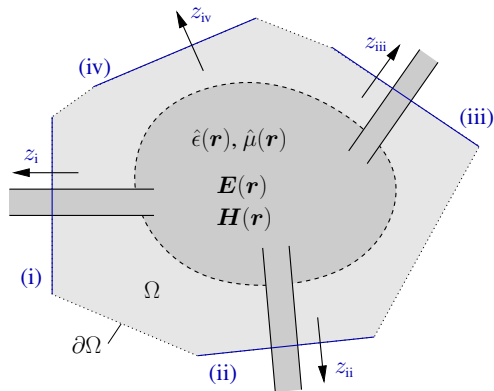


Circuit properties  
for reversed  
wave propagation ?

$$S_{\nu\mu} \longleftrightarrow S_{\mu\nu} ?$$

## Scattering matrices, reciprocity

$\sim \exp(i\omega t)$  (FD)



Circuit properties  
for reversed  
wave propagation ?

$$S_{\nu\mu} \longleftrightarrow S_{\mu\nu} ?$$

- $\mathbf{E}_1, \mathbf{H}_1$  and  $\mathbf{E}_2, \mathbf{H}_2$  solve  $\nabla \times \mathbf{E} = -i\omega\mu_0\hat{\mu}\mathbf{H}$ ,  $\nabla \times \mathbf{H} = i\omega\epsilon_0\hat{\epsilon}\mathbf{E}$ .

$$\hookrightarrow \nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 + \mathbf{H}_1 \times \mathbf{E}_2) = 0, \quad \text{if } \hat{\epsilon} \text{ and } \hat{\mu} \text{ are symmetric.}$$

(i.e. if  $\hat{\epsilon}^T = \hat{\epsilon}$ ,  $\hat{\mu}^T = \hat{\mu}$ .)

(Note: order of factors, no complex conjugates.)

## Scattering matrices, reciprocity

- $\mathbf{E}_1, \mathbf{H}_1$  and  $\mathbf{E}_2, \mathbf{H}_2$  solve  $\nabla \times \mathbf{E} = -i\omega\mu_0\hat{\mu}\mathbf{H}$ ,  $\nabla \times \mathbf{H} = i\omega\epsilon_0\hat{\epsilon}\mathbf{E}$   
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↳  $\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 + \mathbf{H}_1 \times \mathbf{E}_2) = 0$ , if  $\hat{\epsilon}$  and  $\hat{\mu}$  are symmetric,

↳  $0 = \int_{\Omega} \nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 + \mathbf{H}_1 \times \mathbf{E}_2) d^3r = \int_{\partial\Omega} (\mathbf{E}_1 \times \mathbf{H}_2 + \mathbf{H}_1 \times \mathbf{E}_2) \cdot d\mathbf{a}.$

## Scattering matrices, reciprocity

- $\mathbf{E}_1, \mathbf{H}_1$  and  $\mathbf{E}_2, \mathbf{H}_2$  solve  $\nabla \times \mathbf{E} = -i\omega\mu_0\hat{\mu}\mathbf{H}$ ,  $\nabla \times \mathbf{H} = i\omega\epsilon_0\hat{\epsilon}\mathbf{E}$

↳  $\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 + \mathbf{H}_1 \times \mathbf{E}_2) = 0$ , if  $\hat{\epsilon}$  and  $\hat{\mu}$  are symmetric,

↳  $0 = \int_{\Omega} \nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 + \mathbf{H}_1 \times \mathbf{E}_2) d^3r = \int_{\partial\Omega} (\mathbf{E}_1 \times \mathbf{H}_2 + \mathbf{H}_1 \times \mathbf{E}_2) \cdot d\mathbf{a}.$

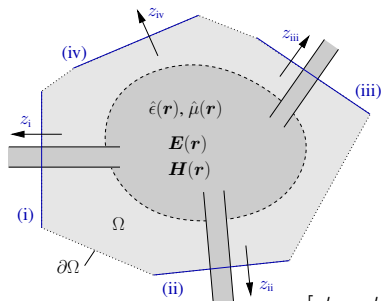
- Fields on  $\partial\Omega$ :  $\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}_j = \sum_{\nu \in \mathcal{N}} \{F_{j,\nu}\psi_{\nu}^f + B_{j,\nu}\psi_{\nu}^b\}, \quad j = 1, 2,$

$$[\psi_a; \psi_b] := \int_{\partial\Omega} (\mathbf{E}_a \times \mathbf{H}_b + \mathbf{H}_a \times \mathbf{E}_b) \cdot d\mathbf{a},$$

↳  $0 = \sum_{\nu} \sum_{\mu} ( F_{1,\nu}F_{2,\mu}[\psi_{\nu}^f; \psi_{\mu}^f] + F_{1,\nu}B_{2,\mu}[\psi_{\nu}^f; \psi_{\mu}^b] \\ + B_{1,\nu}F_{2,\mu}[\psi_{\nu}^b; \psi_{\mu}^f] + B_{1,\nu}B_{2,\mu}[\psi_{\nu}^b; \psi_{\mu}^b] ).$

## Scattering matrices, reciprocity

$\sim \exp(i\omega t)$  (FD)



$$[\boldsymbol{\psi}_a; \boldsymbol{\psi}_b] := \int_{\partial\Omega} (\mathbf{E}_a \times \mathbf{H}_b + \mathbf{H}_a \times \mathbf{E}_b) \cdot \mathbf{d}\mathbf{a}.$$

- $[\boldsymbol{\psi}_\nu; \boldsymbol{\psi}_\mu] = 0$ , if  $\nu$  and  $\mu$  relate to different ports.
- If  $\nu$  and  $\mu$  relate to the same port plane  $p$ :

$$[\boldsymbol{\psi}_\nu^r; \boldsymbol{\psi}_\mu^d] = \iint_p (E_{\nu x}^r H_{\mu y}^d - E_{\nu y}^r H_{\mu x}^d - H_{\nu y}^r E_{\mu x}^d + H_{\nu x}^r E_{\mu y}^d) dx_p dy_p.$$

## Scattering matrices, reciprocity

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- Compare with the modal orthogonality relations on port plane  $p$ ,

$$(\psi_\nu^r; \psi_\mu^d) = \frac{1}{4} \iint_p ((E_{\nu x}^r)^* H_{\mu y}^d - (E_{\nu y}^r)^* H_{\mu x}^d + (H_{\nu y}^r)^* E_{\mu x}^d - (H_{\nu x}^r)^* E_{\mu y}^d) dx_p dy_p.$$

## Scattering matrices, reciprocity

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- Compare with the modal orthogonality relations on port plane  $p$ , for propagating modes with real transverse components:

$$(\psi_\nu^r; \psi_\mu^d) = \frac{1}{4} \iint_p (E_{\nu x}^r H_{\mu y}^d - E_{\nu y}^r H_{\mu x}^d + H_{\nu y}^r E_{\mu x}^d - H_{\nu x}^r E_{\mu y}^d) dx_p dy_p,$$

$$(\psi_\nu^f; \psi_\mu^f) = \delta_{\nu\mu} P_\nu, \quad (\psi_\nu^b; \psi_\mu^b) = -\delta_{\nu\mu} P_\nu, \quad (\psi_\nu^f; \psi_\mu^b) = (\psi_\nu^b; \psi_\mu^f) = 0.$$

## Scattering matrices, reciprocity

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- $\psi^f = (E_x, E_y, iE_z, H_x, H_y, iH_z)^\top$   
 $\psi^b = (E_x, E_y, -iE_z, -H_x, -H_y, iH_z)^\top$ . (Real components).

## Scattering matrices, reciprocity

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$$[\psi_\nu^f; \psi_\mu^f] = [\psi_\nu^b; \psi_\mu^b] = 0, \quad [\psi_\nu^f; \psi_\mu^b] = -\delta_{\nu\mu} 4P_\nu, \quad [\psi_\nu^b; \psi_\mu^f] = \delta_{\nu\mu} 4P_\nu.$$

## Scattering matrices, reciprocity

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$$\hookrightarrow 0 = \sum_{\nu} 4P_{\nu} (B_{1,\nu} F_{2,\nu} - F_{1,\nu} B_{2,\nu}),$$

uniform normalization  $P_{\nu} = P_0$ ,

$$\hookrightarrow 0 = \sum_{\nu} (B_{1,\nu} F_{2,\nu} - F_{1,\nu} B_{2,\nu}),$$

$$\hookrightarrow 0 = \mathbf{B}_1 \cdot \mathbf{F}_2 - \mathbf{F}_1 \cdot \mathbf{B}_2,$$

$$\mathbf{F}_j = \mathbf{S} \mathbf{B}_j,$$

$$\hookrightarrow 0 = \mathbf{B}_1 \cdot \mathbf{S} \mathbf{B}_2 - (\mathbf{S} \mathbf{B}_1) \cdot \mathbf{B}_2,$$

$$\hookrightarrow 0 = \mathbf{B}_1 \cdot \mathbf{S} \mathbf{B}_2 - \mathbf{B}_1 \cdot \mathbf{S}^{\top} \mathbf{B}_2,$$

$$\hookrightarrow 0 = \mathbf{B}_1 \cdot (\mathbf{S} - \mathbf{S}^{\top}) \mathbf{B}_2 \text{ for all } \mathbf{B}_1, \mathbf{B}_2.$$

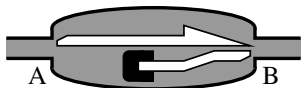
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$$\mathbf{S} = \mathbf{S}^{\top}, \quad S_{\nu\mu} = S_{\mu\nu} \text{ for all } \nu, \mu.$$

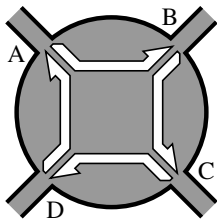
The scattering matrix of a *reciprocal circuit* is *symmetric*.

Reciprocal circuit: made of reciprocal media, with  $\hat{\epsilon} = \hat{\epsilon}^{\top}$ ,  $\hat{\mu} = \hat{\mu}^{\top}$ .

## Nonreciprocal devices



Isolator:  
unidirectional transmission,  
 $S_{BA} = 1, S_{AB} = 0.$



Circulator:  
transmission cycle,  
 $S_{BA} = 1, S_{CB} = 1, S_{DC} = 1, S_{AD} = 1,$   
 $S_{..} = 0$  otherwise.

Required: nonreciprocal media with  $\hat{\epsilon} \neq \hat{\epsilon}^T$ ,  
↔ magneto optic media, Faraday effect.

## *Nonreciprocal devices*

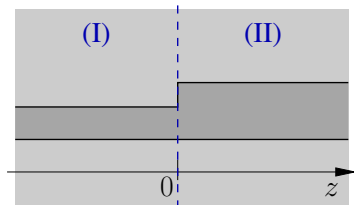
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What about, for example,

- a long, “adiabatic” Y-junction ?
- a junction between a single mode core and a wider multimode waveguide ?

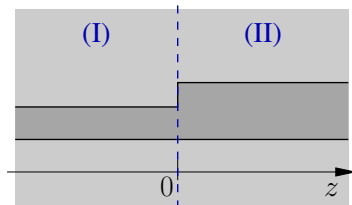


## Waveguide discontinuities



Half-infinite waveguides (I), (II),  
discontinuity at  $z = 0$ .

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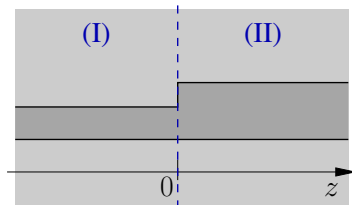
- Expand into local normal modes  
 $\{\psi_{s,m}^d, \beta_{s,m}\}$ ,  $m \in \mathcal{N}_s$ ,  $s = \text{I, II}$ :  
 Transverse boundary conditions  $\leftrightarrow$  discrete sets.

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}_s(x, y, z) = \sum_{m \in \mathcal{N}_s} \left\{ f_{s,m} \psi_{s,m}^f(x, y) e^{-i\beta_{s,m}z} + b_{s,m} \psi_{s,m}^b(x, y) e^{+i\beta_{s,m}z} \right\},$$

$z < 0$ :  $s = \text{I}$ ,  $f_{\text{I},m}$  given influx,  $b_{\text{I},m}$  unknown,  
 $z > 0$ :  $s = \text{II}$ ,  $f_{\text{II},m}$  unknown,  $b_{\text{II},m}$  given influx.

$\hookrightarrow (\mathbf{E}, \mathbf{H})_{\text{I,II}}$  are solutions for  $z < 0$  and  $z > 0$ .

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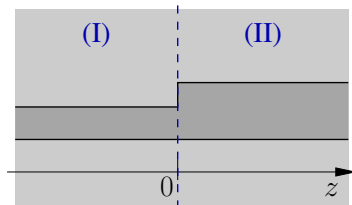
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- Continuity of the tangential components of  $\mathbf{E}, \mathbf{H}$  at the interface  
 $\leftrightarrow$  formally equate expressions for  $(\mathbf{E}, \mathbf{H})_{\text{I,II}}$  at  $z = 0$ .

(Only equality of  $E_x, E_y, H_x, H_y$  will be relevant.)

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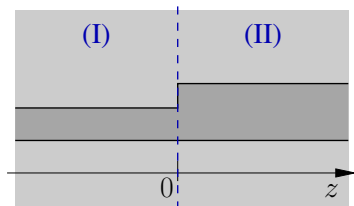
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(Only equality of  $E_x, E_y, H_x, H_y$  will be relevant.)

- Project on  $\psi_{s,l}^d$  to extract coefficients ...

## Waveguide discontinuities, scattering matrix



(Global coordinate  $z \neq$  former local coordinate on port I)  
(One variant of a projection procedure.)

- $(\psi_{I,l}^b; \cdot = \cdot)$ ,  $l \in \mathcal{N}_I$ :

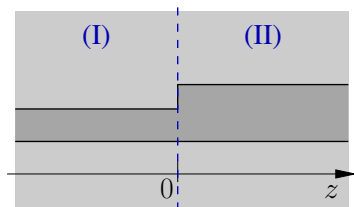
$$\sum_{m \in \mathcal{N}_I} [f_{I,m}(\psi_{I,l}^b; \psi_{I,m}^f) + b_{I,m}(\psi_{I,l}^b; \psi_{I,m}^b)] = \sum_{m \in \mathcal{N}_{II}} [f_{II,m}(\psi_{I,l}^b; \psi_{II,m}^f) + b_{II,m}(\psi_{I,l}^b; \psi_{II,m}^b)],$$

- $(\psi_{II,l}^f; \cdot = \cdot)$ ,  $l \in \mathcal{N}_{II}$ :

$$\sum_{m \in \mathcal{N}_I} [f_{I,m}(\psi_{II,l}^f; \psi_{I,m}^f) + b_{I,m}(\psi_{II,l}^f; \psi_{I,m}^b)] = \sum_{m \in \mathcal{N}_{II}} [f_{II,m}(\psi_{II,l}^f; \psi_{II,m}^f) + b_{II,m}(\psi_{II,l}^f; \psi_{II,m}^b)],$$

$$\left( \begin{array}{c} \mathbf{b}_I \\ \mathbf{f}_I \end{array} \right) = \mathbf{S} \left( \begin{array}{c} \mathbf{f}_I \\ \mathbf{b}_{II} \end{array} \right) = \left( \begin{array}{cc} \mathbf{S}_{I,I} & \mathbf{S}_{I,II} \\ \mathbf{S}_{II,I} & \mathbf{S}_{II,II} \end{array} \right) \left( \begin{array}{c} \mathbf{f}_I \\ \mathbf{b}_{II} \end{array} \right).$$

## Waveguide discontinuities, overlap model



Most simplified variant:  
Unidirectional **overlap model**.

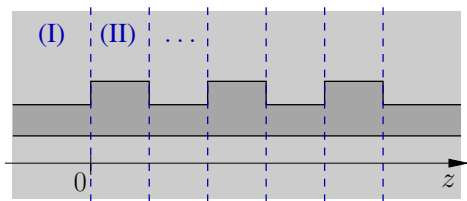
- (I): Incoming guided mode  $\psi_I$ , reflections & radiation neglected.  
(II): Outgoing guided modes  $\psi_{II,m}$ , radiation neglected.

- $f_I \psi_I \approx \sum_m f_{II,m} \psi_{II,m}$  at  $z = 0$ .

↪ 
$$f_{II,m} = \frac{(\psi_{II,m}; \psi_I)}{(\psi_{II,m}; \psi_{II,m})} f_I, \quad \text{or} \quad f_{II,m} = \frac{1}{P_{II,m}} (\psi_{II,m}; \psi_I) f_I.$$

(Transmission is given directly by the “overlaps” ↪ Relevance of the mode products  $(\cdot; \cdot)$ .  
(Cf. explicit expressions for overlaps of 2-D modes, involving only principal mode profile components.)

## A sequence of waveguide discontinuities



- Divide into segments.
- Establish local normal mode expansions.
- Project on local modes.

↪ Linear system of equations for all local mode amplitudes.

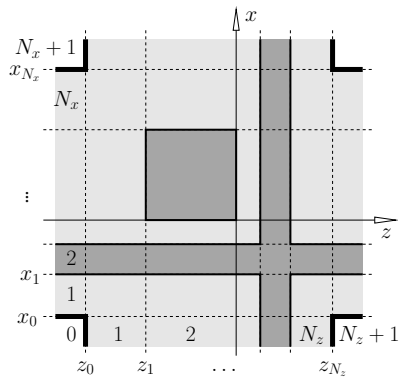
↪ Solve (...)  $\rightsquigarrow$   $\begin{pmatrix} E \\ H \end{pmatrix} (x, y, z)$ .

*Bidirectional eigenmode propagation (BEP),*  
*Eigenmode expansion method (EME),*

...

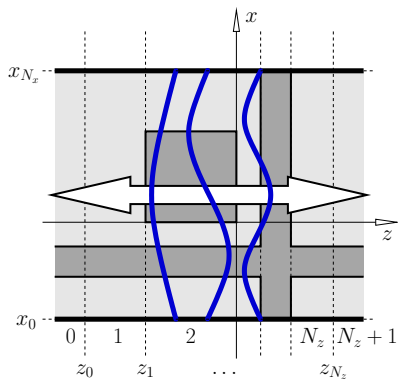
(Radiated outgoing fields: Open boundary conditions required (PMLs)  $\leftrightarrow$  Complex eigenmodes.)  
(2-D: ok. 3-D: ?)

## Rectangular 2-D circuits



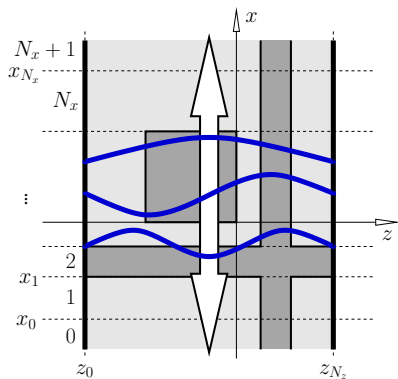
- Divide into slices & layers.

## Rectangular 2-D circuits



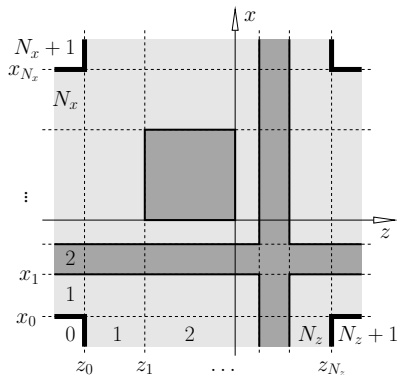
- Divide into slices & layers.
- Establish local modes:  
Propagation along  $\pm z$ ,  
boundary conditions  $\phi = 0$ .

## Rectangular 2-D circuits



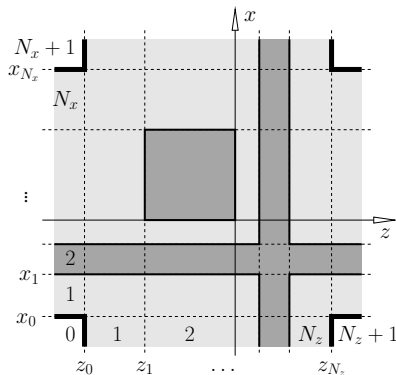
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## Rectangular 2-D circuits

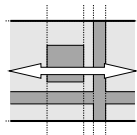


- Divide into slices & layers.
- Establish local modes:
  - Propagation along  $\pm z$ ,
  - & Propagation along  $\pm x$ ,
  - boundary conditions  $\phi = 0$ .
- Project at horizontal & vertical interfaces.

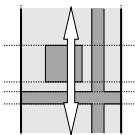
## Rectangular 2-D circuits



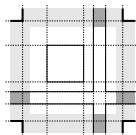
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horizontal BEP,

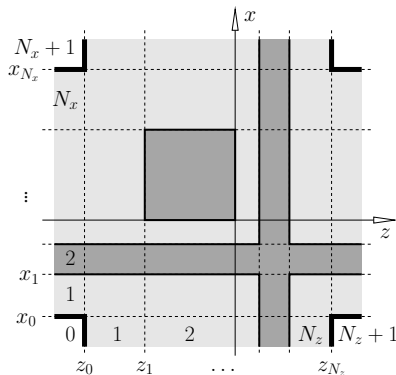


vertical BEP,



continuity at  $x_0, x_N, z_0, z_N$ .

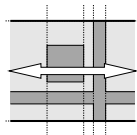
## Rectangular 2-D circuits



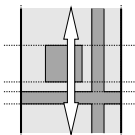
### Quadridirectional Eigenmode Propagation (QUEP)



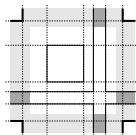
- Divide into slices & layers.
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 Propagation along  $\pm z$ ,  
 & Propagation along  $\pm x$ ,  
 boundary conditions  $\phi = 0$ .
- Project at horizontal  
 & vertical interfaces.



horizontal BEP,



vertical BEP,



continuity at  $x_0, x_N, z_0, z_N$ .

## Upcoming

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Next lectures:

- Bent optical waveguides; whispering gallery resonances; circular microresonators.
- Coupled mode theory, perturbation theory.
- A touch of photonic crystals; a touch of plasmonics.

