

# Optical Waveguide Theory (B)



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## Course overview

### Optical waveguide theory

- A Photonics / integrated optics; theory, motto; phenomena, introductory examples.
- B Brush up on mathematical tools.
- C Maxwell equations, different formulations, interfaces, energy and power flow.
- D Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- E Normal modes of dielectric optical waveguides, mode interference.
- F Examples for dielectric optical waveguides.
- G Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
- H Bent optical waveguides; whispering gallery resonances; circular microresonators.
- I Coupled mode theory, perturbation theory.
  - Hybrid analytical / numerical coupled mode theory.
- J A touch of photonic crystals; a touch of plasmonics.
  - Oblique semi-guided waves: 2-D integrated optics.
  - Summary, concluding remarks.

### Vector calculus, keywords

Ingredients: (here: Cartesian coordinates)

- Space and time coordinates:  $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow (x, y, z), t$ .
- Scalar and vector fields:  $\phi(\mathbf{r}, t), \mathbf{A}(\mathbf{r}, t), \mathbf{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$ .
- Inner product:  $\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$ .
- Vector product:  $\mathbf{A} \times \mathbf{B} = \begin{pmatrix} A_y B_z - A_z B_y \\ A_z B_x - A_x B_z \\ A_x B_y - A_y B_x \end{pmatrix}$ .
- Time derivatives:  $\frac{\partial \phi}{\partial t}, \partial_t \phi, \dot{\phi}, \nabla_t \phi$ .


### Vector calculus, keywords

Ingredients: (here: Cartesian coordinates)

- Del, nabla:  $\nabla = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$ .
- Gradient:  $\text{grad}\phi = \nabla\phi = \begin{pmatrix} \partial_x \phi \\ \partial_y \phi \\ \partial_z \phi \end{pmatrix}$ .
- Divergence:  $\text{div}\mathbf{A} = \nabla \cdot \mathbf{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z$ .
- Curl:  $\text{curl}\mathbf{A} = \text{rot}\mathbf{A} = \nabla \times \mathbf{A} = \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix}$ .
- Laplacian:  $\Delta = \nabla \cdot \nabla = \nabla^2$ ,  
 $\Delta\phi = \partial_x^2 \phi + \partial_y^2 \phi + \partial_z^2 \phi, \Delta\mathbf{A} = \begin{pmatrix} \Delta A_x \\ \Delta A_y \\ \Delta A_z \end{pmatrix}$ .

## Dirac delta

A linear functional

that extracts the value of a function at one point: 

$$1\text{-D: } \int_a^b f(x) \delta(x - x_0) dx = \begin{cases} f(x_0), & \text{if } a < x_0 < b, \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta(x - x_0) = 0, \text{ if } x \neq x_0.$$

$$3\text{-D: } \int_{\mathcal{V}} f(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}_0) d\mathcal{V} = \begin{cases} f(\mathbf{r}_0), & \text{if } \mathbf{r}_0 \in \mathcal{V}, \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta(\mathbf{r} - \mathbf{r}_0) = 0, \text{ if } \mathbf{r} \neq \mathbf{r}_0.$$

Implications: manifold.

## Fourier transform

3-D: A field  $\phi(\mathbf{r})$ :

$$\phi(\mathbf{r}) = \frac{1}{\sqrt{2\pi}^3} \int \tilde{\phi}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}} d^3k, \quad \tilde{\phi}(\mathbf{k}) = \frac{1}{\sqrt{2\pi}^3} \int \phi(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} d^3r.$$

4-D: A field  $\phi(\mathbf{r}, t)$ :

$$\phi(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}^4} \iint \tilde{\phi}(\mathbf{k}, \omega) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} d^3k d\omega,$$

$$\tilde{\phi}(\mathbf{k}, \omega) = \frac{1}{\sqrt{2\pi}^4} \iint \phi(\mathbf{r}, t) e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} d^3r dt.$$

## Fourier transform, 1-D

1-D: A function  $f(x) \in \mathbb{C}$  of one variable:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk, \quad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx.$$

- Arbitrary: positioning of factors  $1/\sqrt{2\pi}$ , signs of exponents.
- $\widetilde{\alpha f_1 + \beta f_2} = \alpha \tilde{f}_1 + \beta \tilde{f}_2$ .
- $f(x) = f(-x) \rightsquigarrow \tilde{f}(k) = \tilde{f}(-k)$ .
- $f(x) = -f(-x) \rightsquigarrow \tilde{f}(k) = -\tilde{f}(-k)$ .
- $f \in \mathbb{R} \rightsquigarrow \tilde{f}(-k) = \tilde{f}^*(k)$ .
- $\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk$ .

## Directionally constant systems

A linear PDE in two unknowns

$$(A \partial_{xx} + B \partial_{yy} + C \partial_{xy} + D \partial_x + E \partial_y + F) \psi(x, y) = 0,$$

coefficients  $A(x, y), \dots, F(x, y)$ .

If the system is constant in  $x$ ,  $\partial_x A = \dots = \partial_x F = 0$ ,

- write  $\psi$  as  $\psi(x, y) = \int \tilde{\psi}(k, y) e^{ikx} dk$ .

$$\int (B \partial_{yy} + (E + ikC) \partial_y + (F + ikD - k^2 A)) \tilde{\psi}(k, y) e^{ikx} dk = 0,$$

$$(B \partial_{yy} + (E + ikC) \partial_y + (F + ikD - k^2 A)) \tilde{\psi}(k, y) = 0, \text{ (for all } k),$$

... a set of DEs in one unknown.

## Directionally constant systems

A linear PDE in two unknowns

$$(A \partial_{xx} + B \partial_{yy} + C \partial_{xy} + D \partial_x + E \partial_y + F) \psi(x, y) = 0,$$

coefficients  $A(x, y), \dots, F(x, y)$ .

If the system is constant in  $x$ ,  $\partial_x A = \dots = \partial_x F = 0$ ,

- use an ansatz  $\psi(x, y) = \tilde{\psi}(y) e^{ikx}$ .

$$\hookrightarrow (B \partial_{yy} + (E + ikC) \partial_y + (F + ikD - k^2 A)) \tilde{\psi}(y) = 0,$$

... a DE in one unknown, with parameter  $k$ .

(& boundary conditions, ...)

## A touch of variational calculus

- **Functional:**  $\mathcal{L} : U \rightarrow \mathbb{R}, \mathbb{C}$ ,  
 $u \rightarrow \mathcal{L}(u)$ ,

a map from a space  $U$  of functions to real / complex numbers.

- **Stationary functional:**  $\left. \frac{d}{ds} \mathcal{L}(u + sv) \right|_{s=0} = 0$  for all  $v$ ,

the variation of  $\mathcal{L}$  at  $u$  vanishes for arbitrary directions  $v$ .

- **Restriction of a functional:**

... to a parametrized family of functions;

- ↔ extremization with respect to these parameters,
- ↔ approximations of stationary points of the functional.

## General solution of the wave equation

$$\left( \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \psi(\mathbf{r}, t) = 0, \quad \psi(\mathbf{r}, 0) = \psi_0(\mathbf{r}), \quad \partial_t \psi(\mathbf{r}, 0) = \phi_0(\mathbf{r}),$$

$$\& \quad \psi(\mathbf{r}, t) = \frac{1}{(2\pi)^2} \iint \tilde{\psi}(\mathbf{k}, \omega) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} d\omega d^3k,$$

$$\hookrightarrow \left( -k^2 + \frac{\omega^2}{c^2} \right) \tilde{\psi}(\mathbf{k}, \omega) = 0,$$

$$\hookrightarrow \tilde{\psi}(\mathbf{k}, \omega) = a_f(\mathbf{k}) \delta(\omega - \omega_k) + a_b(\mathbf{k}) \delta(\omega + \omega_k), \quad \omega_k = c |\mathbf{k}|,$$

$$\hookrightarrow \psi(\mathbf{r}, t) = \frac{1}{(2\pi)^2} \int \left( a_f(\mathbf{k}) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} + a_b(\mathbf{k}) e^{i(\mathbf{k} \cdot \mathbf{r} + \omega_k t)} \right) d^3k,$$

- $\psi(\mathbf{r}, 0) = \psi_0(\mathbf{r}), \partial_t \psi(\mathbf{r}, 0) = \phi_0(\mathbf{r}) \rightsquigarrow \dots \rightsquigarrow a_f(\mathbf{k}), a_b(\mathbf{k})$ .

## A touch of variational calculus

**Example:**

$$U = \{u : [0, \pi] \rightarrow \mathbb{R} \mid u(0) = u(\pi) = 0\},$$

$$\mathcal{L} : U \rightarrow \mathbb{R},$$

$$\mathcal{L}(u) = \frac{\int_0^\pi (\partial_x u)^2 dx}{\int_0^\pi u^2 dx}.$$

(...)

$\mathcal{L}$  stationary at  $u$ ,

$$\left. \frac{d}{ds} \mathcal{L}(u + sv) \right|_{s=0} = 0 \quad \forall v. \quad \longleftrightarrow$$

$u$  satisfies DE & b.c.,

$$\partial_x^2 u = -\lambda u, \quad \lambda = \mathcal{L}(u),$$

$$u(0) = u(\pi) = 0.$$

↓ Restrict  $\mathcal{L}$ ,  $L(\mathbf{a}) = \mathcal{L}(u|\mathbf{a})$ .

$L$  stationary at  $\mathbf{a}$ ,  $\nabla_{\mathbf{a}} L = 0$ .  $\longleftrightarrow$

Approximate solution  
of DE / eigenproblem.

## Upcoming

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Next lectures:

- Maxwell equations, different formulations, interfaces, energy and power flow.
- Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
- Normal modes of dielectric optical waveguides, mode interference.

